Section 7.6:
Laplace Transforms of Discontinuous Functions

Section 7.6: Laplace Transforms of Discontinuous Functions

Discussion

- Why?
- We want to be able to solve DEs of the form $y'' - 3y' + 2y = g(t)$ where $g(t)$ has jump discontinuities
- E.g. Electric Circuits with a switch

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Unit Step Function

Definition 5. The **unit step function** $u(t)$ is defined by

(1) $$u(t) := \begin{cases} 0, & t < 0, \\ 1, & 0 < t. \end{cases}$$

(Any Riemann integral, like the Laplace transform, of a function is unaffected if the integrand's value at a single point is changed by a finite amount. Therefore, we do not specify a value for $u(t)$ at $t = 0$.)

Discussion

- Plug in numbers
- Graph

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Discussion (Piecewise formulas and graphs)

- Shifting the unit step function
- Multiplying the unit step function by a number
- Multiplying the unit step function by a function

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Rectangular Window Function

Definition 6. The **rectangular window function** $\Pi_{a,b}(t)$ is defined by†

(3) $$\Pi_{a,b}(t) := u(t-a) - u(t-b) = \begin{cases} 0, & t < a, \\ 1, & a < t < b, \\ 0, & b < t. \end{cases}$$

Discussion

- Plug in numbers
- Graph
- Why is formula correct?

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Discussion (Piecewise formulas and graphs)

- Multiplying the rectangular window function by a number
- Multiplying the rectangular window function by a function

Any piecewise continuous function can be expressed in terms of window and step functions.

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Example 1 Write the function $f(t) = \begin{cases} 3\,, & t < 2\,, \\ 1\,, & 2 < t < 5\,, \\ t\,, & 5 < t < 8\,, \\ t^2/10\,, & 8 < t \end{cases}$ in terms of window and step functions.

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Note:
Once we know the Laplace Transform of all shifted unit step functions $u(t-a)$, we can find the Laplace Transform of all functions with jump discontinuities

Question: What is $\mathcal{L}\{\, u(t-a)\, \}$? Derive

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Question: What is $\mathcal{L}\{\ \Pi_{a,b}(t)\ \}$? Derive

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The translation property of $F(s)$ discussed in Section 7.3 described the effect on the Laplace transform of multiplying a function by e^{at}. The next theorem illustrates an analogous effect of multiplying the Laplace transform of a function by e^{-as}.

Translation in *t*

Theorem 8. Let $F(s) = \mathcal{L}\{f\}(s)$ exist for $s > \alpha \geq 0$. If a is a positive constant, then

(8) $$\mathcal{L}\{f(t-a)u(t-a)\}(s) = e^{-as}F(s) ,$$

and, conversely, an inverse Laplace transform† of $e^{-as}F(s)$ is given by

(9) $$\mathcal{L}^{-1}\{e^{-as}F(s)\}(t) = f(t-a)u(t-a) .$$

<u>Meaning</u>

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Translation in *t*

Theorem 8. Let $F(s) = \mathscr{L}\{f\}(s)$ exist for $s > \alpha \geq 0$. If a is a positive constant, then

$$\mathscr{L}\{f(t-a)u(t-a)\}(s) = e^{-as}F(s)\,, \qquad (8)$$

and, conversely, an inverse Laplace transform† of $e^{-as}F(s)$ is given by

$$\mathscr{L}^{-1}\{e^{-as}F(s)\}(t) = f(t-a)u(t-a)\,. \qquad (9)$$

Proof:

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In practice it is more common to be faced with the problem of computing the transform of a function expressed as $g(t)u(t-a)$ rather than $f(t-a)u(t-a)$. To compute $\mathcal{L}\{g(t)u(t-a)\}$, we simply identify $g(t)$ with $f(t-a)$ so that $f(t) = g(t+a)$.

$$\mathcal{L}\{g(t)u(t-a)\}(s) = e^{-as}\mathcal{L}\{g(t+a)\}(s)$$

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Example 2 Determine the Laplace transform of $t^2 u(t-1)$

$$\mathcal{L}\{g(t)u(t-a)\}(s) = e^{-as}\mathcal{L}\{g(t+a)\}(s)$$

TABLE 7.1 **Brief Table of Laplace Transforms**

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
1	$\dfrac{1}{s}, \quad s > 0$
e^{at}	$\dfrac{1}{s-a}, \quad s > a$
$t^n, \quad n = 1, 2, \ldots$	$\dfrac{n!}{s^{n+1}}, \quad s > 0$
$\sin bt$	$\dfrac{b}{s^2 + b^2}, \quad s > 0$
$\cos bt$	$\dfrac{s}{s^2 + b^2}, \quad s > 0$
$e^{at}t^n, \quad n = 1, 2, \ldots$	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$
$e^{at}\sin bt$	$\dfrac{b}{(s-a)^2 + b^2}, \quad s > a$
$e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2 + b^2}, \quad s > a$

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Example 3 Determine $\mathcal{L}\{(\cos t)u(t-\pi)\}$

$$\mathcal{L}\{g(t)u(t-a)\}(s) = e^{-as}\mathcal{L}\{g(t+a)\}(s)$$

TABLE 7.1 Brief Table of Laplace Transforms

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
1	$\frac{1}{s}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
$t^n, \quad n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}, \quad s > 0$
$\sin bt$	$\frac{b}{s^2+b^2}, \quad s > 0$
$\cos bt$	$\frac{s}{s^2+b^2}, \quad s > 0$
$e^{at}t^n, \quad n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
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Example 4 Determine $\mathscr{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\}$ and sketch its graph.

$$\mathscr{L}^{-1}\{e^{-as}F(s)\}(t) = f(t-a)u(t-a)$$

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Example 5 The current I in an LC series circuit is governed by the initial value problem

(12) $$I''(t) + 4I(t) = g(t)\,; \qquad I(0) = 0\,, \qquad I'(0) = 0\,,$$

where

$$g(t) := \begin{cases} 1\,, & 0 < t < 1\,, \\ -1\,, & 1 < t < 2\,, \\ 0\,, & 2 < t\,. \end{cases}$$

Determine the current as a function of time t.

$$\mathcal{L}\{g(t)u(t-a)\}(s) = e^{-as}\mathcal{L}\{g(t+a)\}(s)$$

$$\mathcal{L}^{-1}\{e^{-as}F(s)\}(t) = f(t-a)u(t-a)$$

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$$I''(t) + 4I(t) = g(t)\,; \qquad I(0) = 0\,, \qquad I'(0) = 0\,, \tag{12}$$

where

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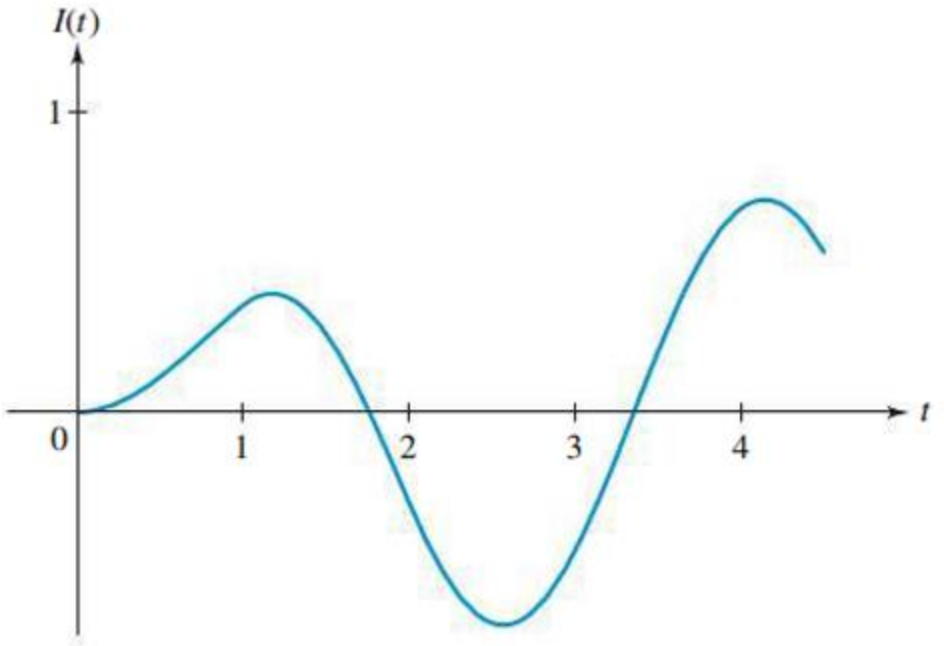

Figure 7.13 Solution to Example 5